Chapter 1

Double And Triple Integrals

1.1 Integral Over An Interval

We start by reviewing integration theory of functions of a single variable.

Given an interval $[a,b]$, a partition P on $[a,b]$ is a collection of points $\{x_j\}$ satisfying $a = x_0 < x_1 < \cdots < x_n = b$. The norm of the partition P, denoted by $\{P\}$, is the maximum of $\Delta x_j = x_j - x_{j-1}, j = 1, \cdots, n$. It measures how fine the partition is. Let f be a function defined on an interval $[a,b]$. The Riemann sum of f with respect to the partition P is defined to be

$$R(f,P) = \sum_{j=1}^{n} f(z_j)\Delta x_j \ ,$$

where the tag z_j is an arbitrary point taken from the subinterval $[x_{j-1}, x_j]$. The Riemann also depends on the choice of tag points, but we simplify things by using the same notation.

The function f is called integrable if there exists a real number α such that for every $\varepsilon > 0$, there is some $\delta > 0$ so that

$$|R(f,P) - \alpha| < \varepsilon, \quad \forall P \ , \|P\| < \delta \ .$$

We call α the integral of f over $[a,b]$ and denote it by

$$\int_a^b f \ , \quad \int_a^b f\, dx \ , \quad \text{or} \ \int_a^b f(x)\, dx \ .$$

When f is non-negative, obviously the Riemann sums are approximate areas and the integral is the area of the set bounded by the x-axis, the graph of f, and the vertical lines $x = a$ and $x = b$.

An immediate question arise: Are there any non-integrable functions? The answer is yes. Let me give you two examples.

First, consider the function $f(x) = 1/x\ , x \in (0, 1]$ and $f(0) = 0$. This is a function defined on $[0, 1]$, which is unbounded near 0. Suppose on the contrary that f is integrable. For $\varepsilon = 1$, there is some δ such that

$$|R(f, P) - \alpha| < 1\ , \quad \forall P,\ \|P\| < \delta.$$

Fix one such P. The inequality $|R(f, P) - \alpha| < 1$ is equivalent to $-1 < R(f, P) - \alpha < 1$. In particular, $R(f, P) - \alpha < 1$, that is, $R(f, P) < 1 + \alpha$, so $f(z_1)\Delta x_1 < 1 + \alpha + \sum_{j=2}^{n} f(z_j)\Delta x_j \equiv \beta$ which is a fixed number. It shows that $1/z_1\Delta x_1 < \beta$. Here β and Δx_1 are fixed number, but the tag point z_1 can be chosen arbitrarily from $(0, 1]$. By choosing it as small as you like, you can make $1/z_1\Delta x_1$ as large as you like, and this contradicts the inequality $1/z_1\Delta x_1 < \beta$. Hence f is not integrable. In fact, it can be shown that all unbounded functions are not integrable.

Second, not all bounded functions are integrable. Consider the function g on $[0, 1]$ defined by $g(x) = 0$ if x is irrational and $g(x) = 1$ if x is rational. g is a function bounded between 0 and 1. As there are rational and irrational numbers in any interval, for each partition P, when we pick a rational number z_j from $[x_j, x_{j+1}]$ to form a tagged partition, the Riemann sum $R(g, P) = \sum_j g(z_j)\Delta x_j = \sum_j \Delta x_j = 1$. On the other hand, picking tag points w_j to be irrational instead, $g(w_j) = 0$ so $R(g, P) = \sum_j g(w_j)\Delta x_j = 0$. You can see that by choosing different tags, the Riemann sums equal to 1 or 0. It cannot converge to a single number α.

Fortunately, most functions people encountered in applications are integrable. It suffices to know that all continuous functions are integrable. In fact, all functions with jump discontinuity are also integrable.

Coming to the evaluation of an integral, from the definition of integrability we have the following approach, namely, take a sequence of tagged partitions $\{P_n\}$ whose norms tend to 0, then

$$\int_a^b f\,dx = \lim_{n\to\infty} R(f, P_n)\ .$$

Although looking very simple, this method is not practical since it involves a limit process which becomes quite complicated even for very simple functions. You may try it on the functions $f(x) = x^2$ or $\sin x$. Now, we are thankful to Issac Newton for his discovery that the evaluation of an integral can be achieved by the following scheme. First, call a function F a primitive function for a given function f if F is differentiable and its derivative is equal to f, that is, $F' = f$. When f is integrable, Newton's fundamental theorem of calculus asserts that

$$\int_a^b f\,dx = F(b) - F(a)\ .$$

As a result, using the simple fact that a primitive function of x^2 is $x^3/3$,

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}\ .$$

Likewise, a primitive function of $\sin x$ is given by $-\cos x$, hence

$$\int_a^b \sin x \, dx = \cos a - \cos b \ .$$

Integrals that had been troubled people since the ancient times are evaluated in this way.

1.2 Double Integral in an Rectangle

Now we come to the integration of functions of two variables. This is a direct extension of what we did in the single variable case where now an interval is replaced by a rectangle.

Let $R = [a, b] \times [c, d]$ be a rectangle and f a bounded function defined in R. Likewise, here a finite set of points

$$\{(x_i, y_j) : \ a = x_0 < x_1 < \cdots < x_n = b, \ c = y_0 < y_1 < \cdots < y_m = d, \}$$

is called a *partition on* R. We denote

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \ ,$$

$$\Delta x_i = x_{i+1} - x_i \ , \quad \Delta y_j = y_{j+1} - y_j \ ,$$

and let $\|P\|$ be the maximum of all $\Delta x_i, \Delta y_j$'s. Pick a point p_{ij} from R_{ij} for each (i, j) we form a collection of tags. A partition together with a choice of tags is called a *tagged partition.*

Let f be a function defined in R. Associate to each tagged partition $(P, \{p_{ij})$, we form a *Riemann sum*

$$R(f, P) = \sum_{i,j} f(p_{ij})|R_{ij}| \ ,$$

where $|R_{ij}| = \Delta x_i \Delta y_j$. A function is called (*Riemann*) *integrable* if there exists a number α such that, for each $\varepsilon > 0$, there is some $\delta > 0$ so that

$$|R(f, P) - \alpha| < \varepsilon \ , \quad \forall P, \ \|P\| < \delta \ .$$

The number α is called the (*Riemann*) *integral* of f over R and is usually denoted by

$$\iint_R f \, dA \ , \quad \text{or} \ \iint_R f \, dA(x, y) \ , \ \text{or} \ \iint_R f(x, y) \, dA(x, y) \ .$$

When f is nonnegative, the Riemann sums are approximate volume and the Riemann integral is the volume of the solid formed between the graph $z = f(x, y)$ and the xy-plane over R.

Just as in the single variable case, unbounded functions are not integrable. In the following discussion, it is implicitly assumed all functions in concern are bounded.

Using the definition of Riemann integral, it is routine to verify that the following basic properties hold:

Theorem 1.1. *Let f and g be integrable in R. For $\alpha, \beta \in \mathbb{R}$.*

1. *$\alpha f + \beta g$ is integrable and*
$$\iint_R (\alpha f + \beta g)\, dA = \alpha \iint_R f\, dA + \beta \iint_R g\, dA\ .$$
2. *fg is integrable.*
3. *f/g is integrable provided that $|g| \geq C$ for some positive constant C.*
4. $$\iint_R f\, dA \geq 0\ ,$$
whenever f is non-negative.

The first property shows that all integrable functions form a real vector space and the mapping
$$f \mapsto \iint_R f\, dA$$
is a linear mapping from this vector space to the space of real numbers. The second and third properties show how nice the integration interacts with algebraic operations of functions.

The fourth property, which may be termed as positivity preserving (or more precisely non-negativity preserving), is an essential one. Note that $f \geq 0$ and $\iint_R f\, dA = 0$ do not necessarily implies $f \equiv 0$. It suffices to observe that a nonnegative function which vanishes everywhere except at finitely points satisfy these two conditions. On the other hand, it is true that they imply $f \equiv 0$ when f is continuous.

Combining linearity and positivity preserving, we have
$$\iint_R g\, dA \geq \iint f\, dA\ ,$$
provided $g \geq f$ in R.

Theorem 1.2. 1. *The constant function c is integrable and*
$$\iint_R c\, dA = c|R|\ , \quad |R| \equiv (b-a)(d-c)\ .$$

2. *There are non-integrable functions in each rectangle.*

3. *Every continuous function is integrable.*

(a) is easily proved. (b) can be shown by considering the function $\varphi(x, y) = 0$ if x is a rational number in $[a, b]$ and $\varphi(x, y) = 1$ when x is irrational. Since there are rational and irrational points in each subrectangle R_{ij}, by choosing suitable tags, $\varphi(p_{ij})$ could be 0 or 1. Consequently, each $f(p_{ij})|R_{ij}|$ is either equal to 0 or $|R_{ij}|$. It follows that the Riemann sum of the same partition could be 0 or $\sum_{i,j} |R_{ij}| = (b - a)(d - c)$. It is impossible to find a number α such that $|R(f, P) - \alpha| < \varepsilon$ for all tags.

We will not prove (c), but simply point out that it is based on a fundamental result, which will be used later.

Theorem 1.3. ***(Uniform Continuity Theorem)*** *Every continuous function in a region R satisfies the following property: Given $\varepsilon > 0$, there is some $\delta > 0$ such that*

$$|f(x, y) - f(x', y')| < \varepsilon \ ,$$

for all $(x, y), (x', y') \in R$, $\sqrt{(x - x')^2 + (y - y')^2} < \delta$.

Here our concern is how to evaluate a double integral. Thankfully we have the following result which reduces it to an iterated integral (two integrals of a single variable). We do not need a new version of the fundamental theorem of calculus.

Theorem 1.4. ***(Fubini's Theorem)*** *Let f be a continuous function in R. Then*

$$\iint_R f \, dA = \int_a^b \int_c^d f(x, y) \, dy dx \ .$$

The idea is simple. The double integral can be approximated by Riemann sums. Taking tags of the form (x_i^*, y_j^*), we have

$$\iint_R f \, dA \approx \sum_{i,j} f(x_i^*, y_j^*) \Delta x_i \Delta y_j = \sum_i \left(\sum_j f(x_i^*, y_j^*) \Delta y_j \right) \Delta x_i \ .$$

When $\|P\|$ is very small, both Δy_j and Δx_i are also very small,

$$\sum_i \left(\sum_j f(x_i^*, y_j^*) \Delta y_j \right) \Delta x_i \ \approx \sum_i \int_c^d f(x_i^*, y) \, dy \ \Delta x_i \approx \int_a^b \left(\int_c^d f(x, y) dy \right) dx \ .$$

A similar result holds when the role of x and y are switched. In other words,

$$\iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx dy \ .$$

It implies the “commutative relation”

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy\ .$$

Example 1.1 Evaluate

$$\iint_R xy^2\,dA\ ,$$

where R is the rectangle $[0,1]\times[0,2]$. By Fubini's Theorem,

$$\begin{aligned}\iint_R xy^2\,dA &= \int_0^2 \int_0^1 xy^2\,dy\,dx \\ &= \int_0^2 \frac{xy^3}{3}\Big|_{y=0}^{y=1} dx \\ &= \int_0^2 \frac{x}{3}\,dx \\ &= \frac{2}{3}\ .\end{aligned}$$

Alternatively,

$$\begin{aligned}\iint_R xy^2\,dA &= \int_0^1 \int_0^2 xy^2\,dx\,dy \\ &= \int_0^1 \frac{x^2y^2}{2}\Big|_{x=0}^{x=2} dy \\ &= \int_0^1 2y^2\,dy \\ &= \frac{2}{3}\ .\end{aligned}$$

Sometimes, the order of integration matters.

Example 1.2 Evaluate

$$\iint_R x\sin xy\ dA\ ,$$

where $R=[0,1]\times[0,\pi]$.

We have

$$\begin{aligned}\iint_R x\sin xy\,dA &= \int_0^\pi \int_0^1 x\sin xy\,dxdy \\ &= \int_0^\pi \left(\frac{-\cos y}{y}+\frac{\sin y}{y^2}\right)\,dy\ .\end{aligned}$$

At this point we don't know how to proceed further. So we change the order of integration.

$$\begin{aligned}\iint_R x\sin xy\,dA &= \int_0^1\int_0^\pi x\sin xy\,dy\,dx \\ &= \int_0^1 x\times\frac{-\cos xy}{x}\Big|_{y=0}^{y=\pi}dx \\ &= \int_0^{x^2} 0\,dy + \int_0^1(-\cos\pi x+1)\,dx \\ &= 1\ .\end{aligned}$$

Example 1.3 Let f be the function in $R=[-1,1]\times[0,1]$ given by $f(x,y)=2,\ x^2<y$ and $f(x,y)=0,\ x^2>y$. Evaluate

$$\iint_R x^2 f(x,y)\,dA\ .$$

By Fubini's Theorem,

$$\iint_R x^2 f(x,y)\,dA=\int_{-1}^1\int_0^1 x^2 f(x,y)\,dydx\ .$$

As

$$\begin{aligned}\int_0^1 f(x,y)\,dy &= \int_0^{x^2} f(x,y)\,dy+\int_{x^2}^1 f(x,y)\,dy \\ &= \int_{x^2}^1 2\,dy \\ &= 2(1-x^2)\ ,\end{aligned}$$

we have

$$\iint_R x^2 f(x,y)\,dA=\int_{-1}^1 x^2\times 2(1-x^2)\,dx=2\left(\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_{-1}^1=\frac{8}{15}\ .$$

1.3 Regions In The Plane

First of all, a C^1-curve is a curve that admits a tangent at every point and the tangent changes continuously as the points vary. In a suitable coordinates, the curve can be locally expressed as the graph $(x,f(x))$ of a C^1-function, that is, a function whose derivative exists and is continuous. A curve is simple if it has no self-intersection point. It is closed if it closes up and has no endpoints. Intuitively speaking, a simple closed curve looks like a deformed circle. We will also consider piecewise C^1-curves, that is, those continuous curves which are C^1 except at finitely many points. A set which is bounded by one or several closed piecewise C^1-curves is called a *region* or a *domain*. This definition is not consistent with the usual definition of a region in mathematics literature. However, we will adopt this definition by following our textbook.

Here are some examples of regions.

- $D_r = \{(x,y): \ x^2+y^2 < r^2\}$ is the disk, the region bounded by the unit circle which is a simple closed C^1-curve.
- $\{(x,y): \ x^2/a^2 + y^2/b^2 = 1\}$. The ellipse is also a simple closed C^1-curve which bounds a region.
- Let C_1 and C_2 be two circles with C_1 contained in C_2. These two circles bound a region. The punctured disk $D_r \setminus \{(0,0)\}$ is also a region where the point $\{(0,0)\}$ may be viewed as a degenerate circle.
- Let Δ be the points lying on or inside a triangle. A triangle is a simple, closed, piecewise C^1-curve composed of three line segments. Tangents do not exist at the three vertices.
- Similarly, every polygon whose boundary is a simple, closed piecewise C^1-curve is a region.
- The cardioid $\{(r,\theta): \ r = 1+\cos\theta)\}$ (in polar coordinates) is a simple closed, piecewise C^1-curve which admits a non-differentiable point (ie, a cusp) at the origin. It also bounds a region.

A region must be bounded from its definition. It consists of interior points and boundary points. In this chapter,

A curve always means a simple, piecewise C^1-curve and a region is the plane set bounded by one or several simple, closed piecewise C^1-curves or points.

In Advanced Calculus I, the objects of study are continuous and differentiable functions. In integration theory the classes of functions are wider. Just like we are able to integrate functions with discontinuity jumps in a single variable, we can integrate functions which admit discontinuous points along some curves.

1.4 Double Integral In A Region

Now we consider double integrals over a region D which is not necessarily a rectangle. A quick way to achieve this goal is to extend f which is only defined in D to the entire space by setting it to be zero outside D. We may call $\tilde{f}$ the extension of f from D. By picking a rectangle R containing D, we may simply define

$$\iint_D f dA = \iint_R \tilde{f}\, dA \ ,$$

where $\tilde{f}$ is the extended function of f from D. To justify this approach, we need to clarify two points. The first one is the definition must be independent of the choice of the rectangle. The next one seems more serious. Namely, even if the function f is continuous in R, the extended function $\tilde{f}$ may develop a jump discontinuity across the boundary of D.

Theorem 1.5. *Let R_1 and R_2 be two rectangles containing D in their interior. Then*

$$\iint_{R_1} \tilde{f}\, dA = \iint_{R_2} \tilde{f}\, dA \ ,$$

provided $\tilde{f}$ is integrable in R_1 and R_2.

Proof. Since $D \subset R_1 \cap R_2 \subset R_i, i = 1, 2$, it suffices to show

$$\int_{R_1} \tilde{f}\, dA = \int_{R_3} \tilde{f}\, dA \ ,$$

where $D \subset R_3 \subset R_1$. But this is obvious as $\tilde{f} \equiv 0$ in $R_1 \setminus R_3$.

□

Theorem 1.6. *Every bounded function in a rectangle R which is continuous except on one or several piecewise C^1-curves is integrable.*

This is a reasonable generalization of the integrability of functions which are piecewise continuous in the single variable case.

Proof. We will give a proof for the special case where f in continuous in the rectangle $R = [a, b] \times [c, d]$ except at a horizonal line $y = \alpha,\ \alpha \in (c, d)$. Given $\varepsilon > 0$, we first fix a small number ρ such that $M\rho(b - a) < \varepsilon/3$ (M is a bound on $|f|$). Next we define a new function f_ρ which is equal to f in $[a, b] \times [c, \alpha - \rho]$ and $[a, b] \times [\alpha + \rho, d]$, and, for linear from $(x, \alpha - \rho)$ to $(x, \alpha + \rho)$. f_ρ is continuous in R and integrable. Now, for the given ε, we can find some δ such that

$$\left| R(f_\rho, P) - \iint_R f_\rho\, dA \right| < \frac{\varepsilon}{2} \ , \quad \forall P, \quad \|P\| < \delta \ .$$

Letting $R_1 = [a,b] \times [c,\alpha]$ and $R_2 = [a,b] \times [\alpha,d]$, we have

$$\begin{aligned}
\left| R(f,P) - \int_a^b \int_c^d f(x,y)\,dydx \right| &\leq |R(f,P) - R(f_\rho,P)| \\
&\quad + \left| R(f_\rho,P) - \int_a^b \int_c^d f_\rho\,dydx \right| \\
&\quad + \left| \int_a^b \int_c^d f_\rho\,dydx - \int_a^b \int_c^d f\,dydx \right| \\
&\equiv A + B + C \ .
\end{aligned}$$

To estimate (A), observing that $R(f,P) - R(f_\rho,p) = \sum_{i,j}(f(z_{ij}) - f_\rho(z_{ij}))|R_{ij}|$ where the summation is over all those subrectangles R_{ij} that touch the strip $[a,b] \times [\alpha-\rho, \alpha+\rho]$. We have

$$A = |R(f,P) - R(f_\rho,p)| \leq \sum_{i,j} |f(z_{ij}) - f_\rho(z_{ij})||R_{ij}| \leq 2M \times (b-a) \times 2(\rho+\delta) \ .$$

Next, by Fubini's Theorem, $B \leq \varepsilon/2$ by our choice of P. Third, since $f_\rho = f$ outside $[a,b] \times [\alpha-\rho, \alpha+\rho]$, we have

$$C \leq 2M(b-a) \times 2\rho \ .$$

Putting these three estimates together, we see that

$$\left| R(f,P) - \int_a^b \int_c^d f\,dydx \right| < \frac{\varepsilon}{2} + 4M(b-a)(\rho+\delta) + 4M(b-a)\rho \ .$$

The left hand side of this estimate is independent of ρ. Letting $\rho \to 0$, we obtain

$$\left| R(f,P) - \int_a^b \int_c^d f\,dydx \right| < \frac{\varepsilon}{2} + 4M(b-a)\delta \ .$$

Now we restrict δ so that $4M(b-a)\delta < \varepsilon/2$, we conclude that

$$\left| R(f,P) - \int_a^b \int_c^d f\,dA \right| < \varepsilon \ ,$$

whenever $\|P\| < \delta$. So f is integrable in R and in fact

$$\iint_R f\,dA = \int_a^b \int_c^d f\,dydx \ .$$

□

Based on these two theorems, we are able to define the integral of a bounded function f over any bounded subset E in $\mathbb{R}^2$ by setting

$$\iint_E f\,dA \equiv \iint_R \tilde{f}\,dA\ , \tag{1.1}$$

where R is any rectangle containing E. The function f is called *integrable* over E provided $\tilde{f}$ is integrable over R. Similar to what is asserted in Theorem 1.5, the integrability of f is independent of the choice of R. When f is nonnegative, the integral of f over D is the volume of the solid bounded between the graph of f and the xy-plane over the region D. When we take $f \equiv 1$, the integral, which becomes

$$\iint_D 1\,dA\ ,$$

reduces to the area of D.

Using (1.1) we have the following extension of Theorem 1.1.

Theorem 1.1'. *Let f and g be integrable in the (bounded) region D. For $\alpha, \beta \in \mathbb{R}$.*

1. *$\alpha f + \beta g$ is integrable in D and*

$$\iint_D (\alpha f + \beta g)\,dA = \alpha \iint_D f\,dA + \beta \iint_D g\,dA\ .$$

2. $$\iint_D f\,dA \geq 0\ ,$$

provided f is non-negative.

In Theorem 1.1 (2) and (3) are concerned with the product and quotient of integrable functions. However, functions appearing in applications are mostly continuous ones. For this reason we do not formulate them in Theorem 1.1'.

Next we show the double integration over a curve vanishes. In other words, a curve is too thin to support a volume.

Theorem 1.7. *Let f be a bounded function on a curve C. Then $\tilde{f}$ is integrable and*

$$\iint_C f\,dA = 0\ .$$

Proof. Let us take $R = [a,b] \times [c,d]$ and consider the special case that $C = \{(x, \varphi(x)\}$ is the graph of a continuous function in $[a,b]$ in R. By Theorem 1.6, $\tilde{f}$ is integrable in R. We have

$$\begin{aligned}
\iint_C f\,dA &= \iint_R \tilde{f}\,dA \quad \text{(by definition)} \\
&= \int_a^b \int_c^d \tilde{f}\,dydx \quad \text{(Fubini's)} \\
&= \int_a^b \int_c^{\varphi(x)} \tilde{f}(x,y)\,dydx + \int_a^b \int_{\varphi(x)}^d \tilde{f}(x,y)\,dydx \\
&= 0\ ,
\end{aligned}$$

since $\tilde{f}(x,y) = 0$ for all y in $[c, \varphi(x))$ and $(\varphi(x), d]$. □

We will associate a set with a function. In this way, sets can be manipulated as functions.

Let E be a nonempty set in $\mathbb{R}^2$ (actually it could be defined in $\mathbb{R}^n$ for any n.) Its *characteristic function* χ_E is defined to be $\chi_E(x,y) = 1\ ,\quad (x,y) \in E$, and $\chi_E(x,y) = 0$ otherwise. Also set $\chi_\phi \equiv 0$. We point out the following relations:

- $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.
- $\chi_{A\cap B} = \chi_A \cdot \chi_B$.
- $\chi_A \le \chi_B$ if and only if $A \subset B$.

Combining the first two, we have

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B\ .$$

We are ready to prove the following frequently used result.

Theorem 1.8. *Divide the region D by a piecewise C^1-curve C to obtain two regions D_1 and D_2. For any integrable function f in D, f is also integrable in $D_i, i = 1, 2$. Moreover,*

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA\ .$$

Proof. Since the boundary of D_i are composed of piecewise C^1-curves, according to Theorem 5, the characteristic functions χ_{D_i} are integrable, so $f\chi_{D_i}$, as product of two integrable functions, is also integrable. From $C = D_1 \cap D_2$ and $\chi_D = \chi_{D_1} + \chi_{D_2} - \chi_{D_1\cap D_2}$, we have $\chi_D = \chi_{D_1} + \chi_{D_2} - \chi_C$. Let R be a rectangle containing D in its interior. We have

$$\begin{aligned}
\iint_D f\,dA &= \iint_R \tilde{f}\,dA \\
&= \iint_R \tilde{f}\chi_{D_1}\,dA + \iint_R \tilde{f}\chi_{D_2} - \iint_R \tilde{f}\chi_C\ .
\end{aligned}$$

The function $\tilde{f}\chi_{D_1}$ is equal to f in D_1 and 0 outside D_1. Therefore, it is the extension of f from D_1, that is,

$$\iint_R \tilde{f}\chi_{D_1}\, dA = \iint_{D_1} f\, dA \ .$$

Similarly, we have

$$\iint_R \tilde{f}\chi_{D_2}\, dA = \iint_{D_2} f\, dA \ ,$$

and

$$\iint_R \tilde{f}\chi_{C}\, dA = \iint_{C} f\, dA \ .$$

Thus,

$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA - \iint_C f\, dA \ ,$$

and the desired formula holds as the last term vanishes according to Theorem 1.7. $\square$

Now we come to evaluation of a double integral in a region. We have discussed how to do it in a rectangle. We will work on two types of special regions. Type I is of the form

$$\{(x,y):\ f_1(x) \le y \le f_2(x),\ a \le x \le b\}\ , \quad f_i,\ i=1,2,\ \text{is continuous}\ ,$$

and Type II is

$$\{(x,y):\ g_1(y) \le x \le g_2(y),\ c \le y \le d\}\ , \quad g_i,\ i=1,2,\ \text{is continuous}\ .$$

More complicated regions could be decomposed to a union of Type I and Type II regions, with the help from Theorem 1.8.

Theorem 1.9. ***(Fubini's Theorem)***

(a) Let D be a Type I region. For a continuous function f in D,

$$\iint_D f(x,y)\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dydx \ .$$

(b) Let D be a Type II region. For a continuous function F in D,

$$\iint_D f(x,y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dxdy \ .$$

Proof. We prove (a) only. Let $R = [a, b] \times [c, d]$ be a rectangle containing the Type I region D. By Theorem 4,

$$\begin{aligned}
\iint_D f(x, y)\, dA &= \iint_R \tilde{f}(x, y)\, dA \\
&= \int_a^b \int_c^d \tilde{f}(x, y)\, dA \\
&= \int_a^b \left(\int_c^{f_1(x)} \tilde{f}(x, y)\, dA + \int_{f_1(x)}^{f_2(x)} \tilde{f}(x, y)\, dA + \int_{f_2(x)}^{d} \tilde{f}(x, y)\, dA \right) \\
&= \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dA.
\end{aligned}$$

□

Example 1.3 Evaluate

$$\iint_D (2y + 1)\, dA \ ,$$

where D is the region bounded by $y = 2x$ and $y = x^2$.

The curves of $y = 2x$ and $y = x^2$ intersect at $(0, 0)$ and $(0, 2)$. The region of integration is expressed as

$$D = \{(x, y) : \ x^2 \leq y \leq 2x, \ x \in [0, 2] \ \} \ .$$

By Fubini's Theorem,

$$\begin{aligned}
\iint_D (2y + 1)\, dA &= \int_0^2 \int_{x^2}^{2x} (2y + 1)\, du\, dx \\
&= \int_0^2 (y^2 + y)\Big|_{x^2}^{2x} dx \\
&= \frac{28}{5} \ .
\end{aligned}$$

The region D can also be expressed as

$$D = \{(x, y) : \ \frac{y}{2} \leq x \leq \sqrt{y}, \ y \in [0, 4]\}.$$

We have

$$\begin{aligned}
\iint_D (2y + 1)\, dA &= \int_0^4 \int_{y/2}^{\sqrt{y}} (2y + 1)\, dx\, dy \\
&= \int_0^4 (2y + 1) \int_{y/2}^{\sqrt{y}} dx\, dy \\
&= \int_0^4 (2y + 1)(\sqrt{y} - \frac{y}{2})\, dy \\
&= \frac{28}{5} \ .
\end{aligned}$$

Example 1.4 Evaluate the iterated integral

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy \ .$$

It is hard to integrate $\sin x/x$, so we switch the order of integration. First, recognize this iterated integral is equal to the double integral

$$\iint_D \frac{\sin x}{x}\, dA \ ,$$

where D is the triangle bounded between $y = 0, y = x$ for $x \in [0, 1]$. By Fubini's Theorem,

$$\begin{aligned}
\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy &= \iint_D \frac{\sin x}{x}\, dA \\
&= \int_0^1 \int_0^x \frac{\sin x}{x}\, dydx \\
&= \int \frac{\sin x}{x} \times x\, dx \\
&= \int_0^1 \sin x\, dx \\
&= 1 - \cos 1 \ .
\end{aligned}$$

Example 1.5 Evaluate the double integral

$$\iint_D x\, dA \ ,$$

where D is the region bounded by $y = 0, x + y = 0$, and the unit circle on the half plane $x \geq 0$.

The line $x + y = 0$ intersection the circle $x^2 + y^2 = 1$ at $(\sqrt{2}/2, -\sqrt{2}/2)$, so D can be described as

$$D = \{(x, y) : \ -y \leq x \leq \sqrt{1 - y^2},\ y \in [-\sqrt{2}/2, 0]\} \ .$$

Hence,

$$\begin{aligned}
\iint_D x\, dA &= \int_{-\sqrt{2}/2}^0 \int_{-y}^{\sqrt{1-y^2}} x\, dxdy \\
&= \frac{1}{2} \int_{-\sqrt{2}/2}^0 (1 - y^2 - y)\, dy \\
&= \frac{\sqrt{2}}{6} \ .
\end{aligned}$$

If one insists to integrate in y first, we observe that D can be expression as the union of D_1 and D_2:
$$D_1 = \{(x,y) : \ 0 \le y \le -x, \ x \in [0, \sqrt{2}/2]\}$$
and
$$D_2 = \{(x,y) : \ -\sqrt{1-x^2} \le y \le 0, \ x \in [\sqrt{2}/2, 1]\} \ .$$
We have
$$\begin{aligned}
\iint_D x\,dA &= \iint_{D_1} x\,dA + \iint_{D_2} x\,dA \\
&= \int_0^{\sqrt{2}/2} \int_{-x}^0 x\,dydx + \int_{\sqrt{2}/2}^1 \int_{-\sqrt{1-x^2}}^0 x\,dydx \\
&= \int_0^{\sqrt{2}/2} x^2\,dx + \int_{\sqrt{2}/2}^1 x\sqrt{1-x^2}\,dx \\
&= \frac{\sqrt{2}}{6} \ .
\end{aligned}$$

Example 1.6 Find the area of the region which is bounded between $y = x^2 - 4, y = x^2 - 1$ and $x \ge 0, y \le 0$.

After sketching the figure, we see that the area of this region D is given by
$$\iint_D 1\,dA = \int_0^1 \int_{x^2-4}^{x^2-1} dydx + \int_1^2 \int_{x^2-4}^0 dydx \ .$$
A straightforward calculation yields
$$\iint_D 1\,dA = \frac{14}{3} \ .$$

As an application of what has been developed, we introduce a definition of the area of a set. Let E be a nonempty set in $\mathbb{R}^2$ (actually in $\mathbb{R}^n$). E is called *rectifiable* if χ_E is integrable. For a rectifiable set E, its area is given by
$$|E| = \iint_E 1\,dA = \iint_R \chi_E\,dA \ , \ E \subset R.$$

We know that every region is rectifiable since its boundary is composed of piecewise C^1-curves. An interesting property is the Euclidean invariant of area. Any Euclidean motion is a composition of translation, rotation and reflection with respect to the x- and y-axes. One can show that the area of a rectifiable set is invariant under any Euclidean motion. This looks like an obvious fact, but can you prove it?